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Algebraic cluster model calculations for vibrational to gamma-unstable shape phase transition in odd-A nuclei

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ABSTRACT

The Algebraic Cluster Model(ACM) is an interacting boson model that gives the relative motion of the cluster configurations in which all vibrational and rotational degrees of freedom are present from the outset. We schemed a solvable extended transitional Hamiltonian based on the SU(1;1) Lie algebra within the framework for two-, three- and four-body algebraic cluster models that explains both regions O(4) U(3), O(7) U(6) and O(10)-U(9), respectively. We suggest that this method can be used to study of k + x nucleon structures with k = 2, 3, 4 and x = 1, 2 and so on. The obtained results in this study confirm that this ACM technique is worth extending for investigating odd-A and odd-odd nuclei. So, the clustering survives the addition of one and two particles. Our studies confirm the importance of the odd nuclei as necessary signatures to characterize the occurrence of the phase transition and to determine the precise position of the critical point.

***Keywords*:** Quantum Phase Transition, Interacting Boson Fermion Model, SU(1,1) algebra, Algebraic Cluster Model (ACM).

I. Introductions

Algebraic models are advantageous in the many-body and in few-body systems. In algebraic models energy eigenvalues and eigenvectors are obtained by diagonalizing a finite-dimensional matrix, rather than by solving a set of coupled differential equations in coordinate space. As an example, we assign the interacting boson model (IBM), which has been very prosperous in the appositive of the collective states in nuclei [1]. Its dynamical symmetries correspond to the quadrupole vibrator, the axially symmetric rotor and the γ-unstable rotor in a geometrical description. In addition to these special solutions, the IBM can describe the intermediate cases between any of them equally well. The first application of the algebraic approach to the few-body systems was the vibron model [2], which was recommended to describe the vibrational and rotational excitations in the diatomic molecules. Algebraic methods give an accurate view to spectroscopic studies and focus on their's symmetries and selection rules to categorize the basis states, and to evaluate matrix elements of physical observables [3]. The binding energy per nucleon for the light nuclei shows large oscillations with the nucleon number with maxima for nuclei with A =4n and Z=N, especially for the nuclei ^{4}He, ^{8}Be, ^{12}C and ^{16}O for n =1, 2, 3 and 4 respectively, which provides a strong indication of the importance of α clustering in these nuclei [4]. The common method is to introduce a $U(\nu+1)$ spectrum generating algebra for a bound-state problem with ν degrees of freedom in which all states are assigned to the symmetric representation $[N]$ of $U(\nu+1)$ [5, 6]. For the $\nu=5$ quadrupole degrees of freedom in collective nuclei this led to

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the introduction of the $U(6)$ interacting boson model [1]. Similarly, the $U(4)$ vibron model was proposed to describe the dynamics of the $\nu = 3$ dipole degrees of freedom of the relative motion of the two objects, e.g. two atoms in a diatomic molecule [2], the two clusters in a nuclear cluster model [7-9], or a quark and antiquark in a meson [10, 11]. An application to the three-body system involves the six degrees of freedom of the two relative vectors which in the algebraic approach leads to a $U(7)$ spectrum generating algebra [12, 13] as an extension of the vibron model. The $U(7)$ model was developed originally to describe the relative motion of the three constituent quarks in baryons [12, 13], but it has also found applications in molecular physics [14, 15] and nuclear physics (^{12}C as a cluster of three a particles) [4, 5]. The algebraic cluster for the four-body systems in terms of a $U(10)$ spectrum-generation algebra was introduced in [15]. An application to the cluster states in ^{16}O suggested that these can be interpreted in terms of rotations and vibrations of tetrahedral configuration of α particles. The triangular configuration in ^{12}C and tetrahedral configuration in ^{16}O implied by the observed rotational sequence, were confirmed by a study of BE(L) electric transitions along the ground state bands [4, 5, 16]. In [17], ^{8}Be, ^{12}C and ^{16}O nuclei were considered by using an infinite-dimensional algebraic method based on the affine $SU(1,1)$ Lie algebra for the transitional descriptions of the vibron model and α-cluster model. The cluster structures with addition of nucleons discussed especially in the Be isotopes with a variety of methods [17-30]. In Ref [31, 32], single-particle levels in cluster potentials in $k\alpha + x$ nucleon structures within the framework of a cluster shell model (CSM) calculated. In nuclear physics s-orbit and p-orbit adjaceny achived by studying in light nuclei as we see in carbon isotopes [18]. Hafstad and Teller studied (4n + 1) nuclei, e.g. ^{9}Be, ^{13}C and ^{17}O. Their ideas were based upon the structure of the ^{5}He nucleus in which the last neutron was in a p-orbit [18, 21, 22].

The study of the quantum phase transitions enjoys a substantial interest in the algebraic models of nuclear structure. There are mutual relations between shapes (phases) and dynamic symmetry limits. The analytical solutions provide a process in which the system undergoes a change from one dynamical symmetry to another one. The first examples [33, 34] were related to the Interaction Boson Model Approximation (IBA) [1] and the Vibron model [1, 3].

The aim of this contribution is to discuss the quantum phase transitions in the algebraic cluster models for the two-, three- and four- body cluster, to transition description in $U(3) \leftrightarrow O(4)$, $U(6) \leftrightarrow O(7)$ and $U(9) \leftrightarrow O(10)$. This model can be solved by using an infinite dimensional algebraic technique in the IBM framework. This method was applied to the $k\alpha + x$ nucleon structures consisting of k α -particles and x nucleons, such as structures ^{9}Be, ^{9}B and ^{10}B, corresponding to the exchange of neutrons and α-particles. In order to describe the phase transition, we calculate some observables such as energy level, level crossing, expectation values of boson number operator and overlap of the ground-state wave function. The results of calculations for these nuclei are presented and are compared with the corresponding experimental data. In this work, the role of a fermion with angular momentum j at the critical point on quantum phase transitions in bosonic systems is investigated.

The specific aims of the present study and the structure of this paper are as follows: In section 2, we introduce the algebraic cluster model, followed by a discussion of the permutation symmetry. Section 3 briefly summarizes the theoretical aspects of the model. Numerical results are presented in section 4 and section 5 is devoted to summarizing and to justifying some conclusions.

II. The Algebraic Cluster Model (ACM)

In this section, we introduce the algebraic cluster model. It is based on the spectrum generating algebra of $U(\nu + 1)$, where $\nu = 3(n - 1)$ represents the number of the relative spatial degrees of freedom. As special cases the ACM

contains the $U(4)$ vibron model for the two-body problems $(n = 2)$, the $U(7)$ model [4, 5, 12, 13, 15] for three-body clusters $(n = 3)$ and the $U(10)$ model [16, 35, 36] for four-body clusters $(n = 4)$. The relevant degrees of freedom of a system of n-body clusters are given by the $n - 1$ relative Jacobi coordinates

$$\vec{\rho}_k = \frac{1}{\sqrt{k(k+1)}}\left(\sum_{i=1}^{k} \vec{r}_i - k\vec{r}_{k+1}\right) \qquad k = 1,2,\ldots,n-1 \tag{1}$$

and their conjugate momenta. Here $\vec{r}_i$ denotes the position vector of the $i - th$ cluster (i=1,2,...,n).

Instead of a formulation in terms of coordinates and momenta, the method of bosonic quantization is used which consists of introducing a dipole boson with $L_P = 1^-$ for each independent relative coordinate and an auxiliary scalar boson with $L_P = 0^+$

$$s^\dagger, b^\dagger_{k,m} \tag{2}$$

with $k = 1,\ldots,n-1$ and $m = -1,0,1$. The scalar boson does not represent an independent degree of freedom, but is added under the restriction that the Hamiltonian commutes with the number operator

$$N = s^\dagger s + \sum_k \sum_m b^\dagger_{k,m} b_{k,m} \tag{3}$$

i.e. the total number of bosons $N = n_s + \sum_k n_k$ is conserved. The set of $\llbracket(3n-2)\rrbracket^2$ bilinear products of creation and annihilation operators spans the Lie algebra of $U(3n-2)$.

In this contribution, we study the ACM for identical clusters which is relevant to α-cluster nuclei such as ^{8}Be , ^{12}C and ^{16}O . For these systems, the Hamiltonian has to be invariant under the permutation group S_n . The permutation symmetry of n identical objects is determined by the transposition $P(12)$ and the cyclic permutation $(12\ldots n)$. All other permutations can be expressed in terms of these two elementary ones. The transformation properties under S_n of all operators in the model originate from those of the building blocks. The scalar boson, $s^\dagger$, transforms as the symmetric representation $[n]$, whereas the dipole Jacobi bosons, $b^\dagger_k$ with $k = 1,\ldots,n-1$ transform as the $n-1$ components of the mixed symmetry representation $[n-1,1]$.

Hamiltonian that describes the relative motion of a system of n identical clusters, and is a scalar under the permutation group S_n and is rotationally invariant. It conserves the parity as well as the total number of bosons, as given by

$$\begin{aligned} H = {} & \epsilon_0 s^\dagger \tilde{s} - \epsilon_1 \sum_i b^\dagger \tilde{b} + u_0 s^\dagger s^\dagger \tilde{s}\tilde{s} \\ & - u_1 \sum_k s^\dagger b^\dagger_k s^\dagger \tilde{b}_k + \nu_0 \Big(\sum_k b^\dagger_k \tilde{b}_k \tilde{s}\tilde{s} \\ & + hc\Big) \\ & + \sum_L \sum_{ijkl} v^{(L)}_{ijkl} [b^\dagger_i \times b^\dagger_j]^{(L)} [\tilde{b}_k \\ & \times \tilde{b}_l]^{(L)} \end{aligned} \tag{4}$$

with $\tilde{b}_{k,m} = (-1)^{(1-m)} b_{k,-m}$ and $\tilde{s} = s$ by construction, the ϵ_0, ϵ_1, u_0, u_1 and ν_0 terms in Equation (4) are invariant under S_n.

In this contribution, we consider two dynamical symmetries of the ACM Hamiltonian for the n-body problem which are related to the group lattice

$$U(3n-2) \supset \{^{U(3n-3)}_{O(3n-2)} \supset O(3n-3)$$

which are called the $U(3n-3)$ and $SO(3n-2)$ limits of the ACM, respectively. A geometric analysis shows that the $U(3n-3)$ limit corresponds for large N to the (an) harmonic oscillator in $3(n-1)$ dimensions and the $SO(3n-2)$ limit to the deformed oscillator in $3(n-1)$ dimensions [35, 36]

III. Theoretical Framework

Algebraic models provide elegant and simple paradigms for the behavior of a wide variety of physical systems. The basic idea of algebraic models is that Hamiltonians and other physical operators of these systems can be realized by using a set of boson operators, since the collective

excitations of these systems can be regarded as a set of interacting bosons. The spectrum of the systems can be generated by an appropriate unitary Lie algebra, called spectrum generating algebra. Dynamical symmetries often play an important role in the approach. There is one to one correspondence between the shapes (phases) and dynamic symmetry limits in which analytical solutions to the model exist. The shape (phase) transition of these models is referred to as a process in which the system undergoes a change from one dynamical symmetry to another one. The method for diagonalization of the Hamiltonian in the transitional region is not as easy as in either of the limits, especially when the dimension of the configuration space is relatively large. To avoid these problems, an algebraic Bethe ansatz method within the framework of an infinite dimensional $SU(1,1)$ Lie algebra has been proposed by Pan et al...[38, 39].

A. The SU(1,1) expression of Bethe ansatz equations for two-cluster systems

1. The odd-A nuclei: $^{9}_{4}Be$ and $^{9}_{4}B$

For more than two decades, is known that the $^{9}_{4}Be$ nucleus is an example of a molecular covalent bond in nuclear physics, where two particles with valence neutron are limited. The $^{9}_{4}Be$ nucleus, which has unlimited system properties $2\alpha + n$ is the "cornerstone" of cluster physics [18, 21, 22, 32]. Due to its low neutron separation threshold, separation of $^{9}_{4}Be$ can be an origination of astable $^{8}_{4}Be$ nuclei. The $^{8}_{4}Be$ isotope is known as the only nucleus whose ground state is distinguished as the α-particle Bose condensate. A study of the division of the $^{9}_{4}Be$ nucleus in α-particle pair appears to be a clear starting point than the more complex $N\alpha$-systems. This method can also be used to describe the odd-A nuclei. For example, $^{9}_{4}Be$ may be assumed to be composed of two α-particles and a valence neutron, forming, at larger $\alpha + \alpha$ separations ^{5}He nuclei, where the neutron abides in a $p_{3/2}$-orbit. We assume for ^{9}Be a structure similar to ^{9}Be, with the odd neutron exchanged by an odd-proton[18, 21, 22, 32].

The boson algebraic structure will be always taken to be $\mathrm{U}^{\mathrm{B}}(4)$, while the fermion algebraic structure will depend on the values of the angular momenta, j, taken into consideration [10]. Two possible dynamical symmetry limits, $U^B(3)$ and $O^B(4)$, are related to the following two algebraic chains,

$$U^B(4) \otimes U^F(2j+1) \supset \begin{Bmatrix} U^{(B)}(3) \\ O^{(B)}(4) \end{Bmatrix} \otimes SU^F(2j+1) \supset O^B(3) \otimes Sp^F(2j+1) \supset O^B(3) \otimes SU^F(2) \supset Spin^{BF}(3) \quad (5)$$

The negative parity states in the odd-mass nuclei $^{9}_{4}Be$ and $^{9}_{4}B$ are built mainly on the $2p_{\frac{3}{2}}$ shell model orbit [40]. The single - particle orbits $1d_{\frac{5}{2}}$ and $2s_{\frac{1}{2}}$ establish the positive parity states in $^{9}_{4}Be$ and $^{9}_{4}B$ isotopes [31]. In this study, a simplifying assumption is made that single particle states are built on the $2p_{\frac{3}{2}}$ and $2s_{\frac{1}{2}}$. The lattices of algebras in these cases are obtained by putting j=3/2 and 1/2 in Eq.(5), respectively.

The Lie algebra corresponding to the $SU(1,1)$ group is generated by the operators S^x where $x = 0$ and ± 1. To extend this model, we introduce the $SU(1,1)$ pairing algebras for s and b bosons as,

$$S^+(s) = \frac{1}{2}s^{\dagger 2} \qquad S^-(s) = \frac{1}{2}s^2$$
$$S^0(s) = \frac{1}{2}(s^\dagger s + \frac{1}{2}) = \frac{1}{2}n_s + \frac{1}{4} \quad (6)$$

$$S^+(b) = \frac{1}{2}b^\dagger . b^\dagger \quad S^-(b) = \frac{1}{2}\tilde{b}.\tilde{b}$$
$$S^0(b) = \tfrac{1}{2}(b^\dagger \tilde{b} + \tfrac{3}{2}) = \tfrac{1}{2}n_b + \tfrac{3}{4} \quad (7)$$

where n_s and n_b are the number operators for s and b bosons which satisfy the following commutation relations

$$[S^0, S^\pm] = \pm S^\pm \qquad [S^+, S^-] = -2S^0 \quad (8)$$

The Casimir operator of $SU(1,1)$ can be written as

$$C_2(SU(1,1)) = S^0(S^0-1) - S^+S^- \quad (9)$$

The representation is determined by a single number k. Let us assume that $|k\mu\rangle$ is a basis vector of an irrep of $SU(1,1)$, where κ can be any positive real number, and $\mu = k, k+1, \ldots$. Then

(10)

$$C_2\big(SU(1,1)\big)|k\mu\rangle = k(k-1)|k\mu\rangle$$
$$S^0|k\mu\rangle = \mu|k\mu\rangle$$

Now, we introduce the operators of infinite dimensional $SU^{sb}(1,1)$ algebra similar to what has been defined by Pan et al. in ref. [40],

$$S_n^{\pm} = c_s^{(2n+1)}S^{\pm}(s) + c_b^{(2n+1)}S^{\pm}(b) \quad (11)$$

$$S_n^0 = c_s^{2n}S^0(s) + c_b^{2n}S^0(b) \quad (12)$$

where c_s and c_b are the real control parameters, and n can be taken to be 1,2,3,… . To evaluate the energy spectra and transition probabilities, let us consider $|lw\rangle$ as the lowest weight state of $SU^{sb}(1,1)$ algebra which should satisfy

$$S^-(s)|lw\rangle = 0 \qquad S^-(b)|lw\rangle = 0 \quad (13)$$

The lowest weight states,$|lw\rangle_{sb}$ are actually a set of basis vectors of the chain $U(4) \supset U(3) \supset O(3) \supset O(2)$ which

(14)

$$|lw\rangle_{sb}^B = |N_B, k_s = \frac{1}{2}(\nu_s + \frac{1}{2}),$$
$$\mu_s = \frac{1}{2}\left(n_s + \frac{1}{2}\right), k_b = \frac{1}{2}\left(L + \frac{3}{2}\right), \mu_b =$$
$$\tfrac{1}{2}(n_b + \tfrac{3}{2}), LM\rangle$$

where $N_B = \nu_s + \nu_b$, $n_b = L$, $n_s = \nu_s = 0$ or 1. Hence, we have

(15)

$$S_n^0|lw\rangle = (c_s^{(2n)}S^{\circ}(s) + c_b^{(2n)}S^{\circ}(b))|lw\rangle = \Lambda_n^0|lw\rangle$$

$$\Lambda_n^0 = (c_s^{(2n)}(\nu_s + \tfrac{1}{2}) + c_b^{(2n)}(L + \tfrac{3}{2}))\tfrac{1}{2} \quad (16)$$

The following Hamiltonian for description of negative and positive states in transitional region is prepared

(17)

$$\hat{H} = gS_0^+S_0^- + \alpha S_1^0 + \beta\hat{C}_2(SO^B(3)) +$$
$$\gamma\hat{C}_2(spin^{BF}(3))$$

For evaluating the eigenvalue of Hamiltonian Eqs. (17), the eigenstates are considered as

$$|k;\nu_s\nu_b n_\Delta LJM\rangle = \Theta S_{x_1}^+S_{x_2}^+S_{x_3}^+\ldots S_{x_k}^+|lw\rangle^{BF} \quad (18)$$

With Clebsch- Gordan (CG) coefficient, we can calculate lowest weight state, $|\mathrm{lw}\rangle^{\mathrm{BF}}$, in terms of boson and fermion part as

(19)

$$|lw\rangle^{BF} = \sum_j \sum_{m_j=-j}^{m_j=+j} C_{m,m-m_j,m_j}^{J,L,j}\,|lw\rangle_{sb}^B|j,m_j\rangle$$

The $C_{m,m_L,m_j}^{J,L,j}$ symbols represent Clebsch-Gordan coefficients.

(20)

$$|lw\rangle_{sb}^B = |N_B, k_s = \frac{1}{2}(\nu_s + \frac{1}{2}), \mu_s =$$
$$\tfrac{1}{2}(n_s + \tfrac{1}{2}), k_b = \tfrac{1}{2}(L + \tfrac{3}{2}), \mu_b = \tfrac{1}{2}(n_b + \tfrac{3}{2}), LM\rangle$$

(21)

$$S_n^0|lw\rangle = (c_s^{(2n)}S^{\circ}(s) + c_b^{(2n)}S^{\circ}(b))|lw\rangle = \Lambda_n^0|lw\rangle$$

(22)

$$\Lambda_n^0 = (c_s^{(2n)}(\nu_s + \tfrac{1}{2}) + c_b^{(2n)}(L + \tfrac{3}{2}))\tfrac{1}{2}$$

The eigenvalues of Hamiltonians Eqs. (17) can then be expressed;

$$E^{(k)} = h^{(k)} + \alpha\Lambda_1^0 + \beta L(L+1) + \gamma J(J+1) \quad (23)$$

Where

(24)

$$h^{(k)} = \sum_{i=1}^{k} \frac{\alpha}{x_i} = \sum_{i=1}^{k} \frac{g c_s^2 (v_s + \frac{1}{2})}{1 - c_s^2 x_i} + \frac{g c_b^2 (L + \frac{3}{2})}{1 - c_b^2 x_i} - \sum_{i \neq j} \frac{2g}{x_i - x_j} \qquad g = 1$$

2. The odd-odd nuclei: $^{10}_{5}B$

The structure of the odd-odd nuclei may be illustrated as an unpaired proton and an unpaired neutron coupled to a boson core. In this paper, the method described in Refs. [38, 39] will be developed and performed to mixed boson-fermion-fermion systems. The approach based on boson-fermion symmetries has been also applied to odd-odd systems. On the other, IBFM has been increased to odd-odd nuclei, and mention to as IBFFM. To simplify computing, the structure of the odd-odd nuclei is described as an unpaired proton and an unpaired neutron coupled to a $^{8}_{4}Be$.

It should be noted that we have investigated the phase transition from rigid to non-rigid shapes in the case that odd proton and odd neutron in j=3/2 configurations coupled to core that undergoes a transition from $U^B(3)$ and $O^B(4)$ condition.

After this, we considered the state that an unpaired proton and an unpaired neutron being in a j=3/2 shell. The algebraic structure underlying our IBFM-1 approach is shown in Eq.(25). The bosons are initially coupled and so are fermions, and then the compounds of bosons and fermions connect to each other. In Eq. (25), the chain upper show the state that bosons have $\mathrm{U^B}(3)$ dynamical symmetric while bosons in chain lower have $\mathrm{O^B}(4)$ dynamical symmetric.

(25)

$$\begin{aligned} U^B(4) \otimes \left(U^{F_\pi}(4) \otimes U^{F_\nu}(4)\right) & \\ \supset \begin{Bmatrix} U^{(B)}(3) \\ O^{(B)}(4) \end{Bmatrix} & \otimes SU^{F_{\pi\nu}}(4) \\ \supset O^B(3) & \otimes Sp^{F_{\pi\nu}}(4) \\ \supset O^B(3) & \otimes SU^{F_{\pi\nu}}(2) \supset O^{BF}_{\pi\nu}(3) \\ \supset O^{BF}_{\pi\nu}(2) & \end{aligned}$$

The Hamiltonian for the odd-odd nuclei may be written as a sum of a boson part and parts describing the residual interaction between boson-fermion and fermion-fermion interaction. The Hamiltonian with $j_\pi = \frac{3}{2}$ and $j_\nu = \frac{3}{2}$ in transitional region between $\mathrm{U}(3) - \mathrm{O}(4)$ limits in terms of the casimir operators of the group chain (Eq. (25)) is prepared

(26)

$$\begin{aligned} \hat{H} = g S^+_{B,0} S^-_{B,0} + \alpha S^0_{B,1} + \beta \hat{C}_2\left(SO^B(3)\right) & \\ + \rho \hat{C}_2\left(SP^{F_{\pi\nu}}(4)\right) & \\ + \delta \hat{C}_2\left(SU^{F_{\pi\nu}}(2)\right) & \\ + \gamma \hat{C}_2\left(O^{BF_{\pi\nu}}(3)\right) & \end{aligned}$$

Eq.(26) is the suggested Hamiltonian for boson - fermion-fermion systems and α , β ,δ, ρ and γ are real parameters. Hamiltonian Eq.(26) is equivalent to Hamiltonian of rigid limit when $c_s = 1$ and with Hamiltonian of non-rigid limit if $c_s = 0$ and $c_b \neq 0$. So, the $c_s \neq c_b \neq 0$ situation just corresponds to transitional region.

For evaluating the eigenvalues of Hamiltonian Eqs. (26) the eigenstates are considered as [36, 37]

(27)

$$\begin{aligned} |k; v_s, v_b, (\xi_1, \xi_2), SL, JM\rangle & \\ = \Theta S^+_{x_1} S^+_{x_2} S^+_{x_3} \ldots S^+_{x_k} |lw\rangle^{BF_{\pi\nu}} & \end{aligned}$$

$N_B, v_b, (\xi_1, \xi_2), S, L, JM$ are quantum numbers of $U^B(4)$, $O^B(4)$, $SP^{F_{\pi\nu}}(4)$, $SU^{F_{\pi\nu}}(2)$, $SO^B(3)$, $O^{BF_{\pi\nu}}(3)$ and $O^{BF_{\pi\nu}}(2)$, respectively.

The lowest weight state $|\mathrm{lw}\rangle^{\mathrm{BF}_{\pi\nu}}$ is calculated as:

(28)

$$\begin{aligned} |lw\rangle^{BF_{\pi\nu}} = \sum_{m_\pi m_\nu m_{\pi\nu} M} & C^{J_{\pi\nu}, j_\pi, j_\nu}_{m_{\pi\nu}, m_\pi, m_\nu} C^{J, J_{\pi\nu}, L}_{M, m_{\pi\nu}, M_b} |j_\pi, m_\pi\rangle \\ & |j_\nu, m_\nu\rangle |lw\rangle^B_{sb} \end{aligned}$$

The eigenvalues of Hamiltonian Eq. (26) can then be expressed;

(29)

$$\begin{aligned} E^{(k)} = h^{(k)} + \alpha \Lambda^0_1 + \beta L(L+1) & \\ + \rho\left(\xi_{1\pi\nu}(\xi_{1\pi\nu} + 3)\right. & \\ \left. + \xi_{2\pi\nu}(\xi_{2\pi\nu} + 1)\right) & \\ + \delta S(S+1) + \gamma J(J+1)) & \end{aligned}$$

B. The Fitting Procedure

In order to obtain the numerical results for energy spectra ($E^{(k)}$) of the considered nuclei, a set of non-linear Bethe-Ansatz equations (BAE) with k- unknowns for k-pair excitations must be solved. To achieve this aim, we have changed variables in two-cluster nuclei as

$$C = \frac{c_s}{c_b} \leq 1, y_i = c_b^2 x_i$$

In addition, the constants of Hamiltonian with the least square fitting processes to experimental data are obtained. A useful and simple numerical algorithm for solving the BAE Equations (24) and for extracting of the constants in comparison with the experimental energy spectra of the considered nuclei is based on using of Matlab software which will be outlined simultaneously. To determine the roots of the BAE with the specified values of ν_s and ν_b, we solve Equation (24) with the definite values of C and α for $i = 1$ and then we use the function "syms var" in Matlab to obtain all roots. We then repeat this procedure with different C and α to minimize the root mean square deviation, σ, between the calculated energy spectra and the experimental counterparts which explore the quality of the extraction processes. The deviation is defined by the equality

$$\sigma = \left(\frac{1}{N_{tot}} |E_{exp}(i) - E_{cal}(i)|^2\right)^{\frac{1}{2}} \quad (30)$$

N_{tot} is the number of energy levels which are included in the extraction processes. We have extracted the best set of Hamiltonian's parameters, i.e. g, α and β, via the available experimental data.

IV. Numerical results

This section presents the results of the numerical solution of the phase transition observable of the algebraic cluster model for the two-, three- and four- body clusters such as level crossing, expectation values of the boson numbers and calculated variation behavior of the overlap of the ground-state wave function. In this research paper, we have taken ^9Be,^9B,^{10}B ; ^{13}C, ^{13}N, ^{14}N; ^{17}O, ^{17}F nuclei for the two-, three- and four- cluster.

A. Energy spectrum and level crossing

In the wake of the theoretical method achieved beforehand, we apply our algebraic model for the cluster model to the^9Be,^9B and ^{10}B nuclei.

In our calculation, we have proposed the control parameters C values in the 0-1 region for the two-, three- and four- body clusters. So, we have analyzed the properties of the ^9Be, ^9B and ^{10}B nuclei in order to investigate the ground- and excited-state spectra related to the models-the best fit which guarantees that the parameters are well determined. Eigenvalues of these models are obtained by solving Bethe Ansatz equations with the extraction processes to experimental data [41-43] to obtain constants of the Hamiltonian. We explore the best-fitting parameters, which are extracted by the procedures explained in Sect. 3 and the least-square fit to the available experimental data for the excitation energies for the ^9Be,^9B and ^{10}B nuclei and the ability of the $SU(1,1)$ -based transitional Hamiltonian in the reproduction of all considered levels and also the acceptable degree of the extraction procedures. To display how the energy levels change as a function of the control parameter C, the lowest energy levels as a function of C f for the ^9Be,^9B and ^{10}Bnuclei are shown in Fig. 1. The root mean square deviation, σ, between the calculated energy spectra and the experimental counterparts as a function of the control parameter C for these nuclei are shown in Fig. 2. Our results show that two-cluster nuclei have vibrational features but the gamma-unstable rotor character is dominant while a dominancy of dynamical symmetry O(7) exist for three-cluster nuclei, and the four-cluster nuclei have dominant vibrational features. We see from the figure that in this case the odd particle drives the system toward deformation or sphericity.

The figures show how the energy levels as a function of the control parameter C evolve from one dynamical symmetry limit to the other. It can be seen from the figures that numerous level crossings occur.

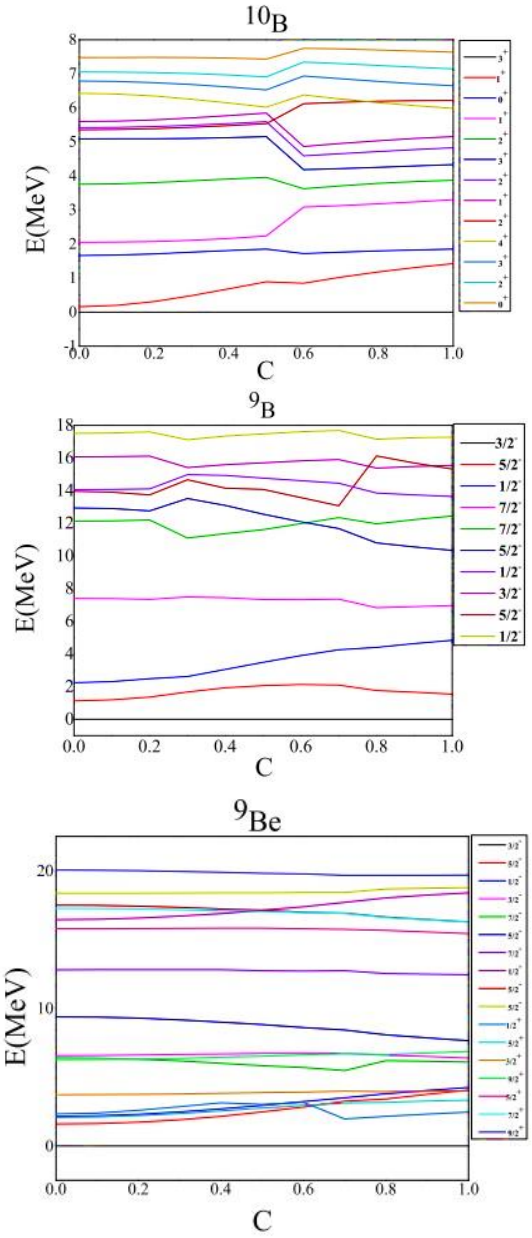
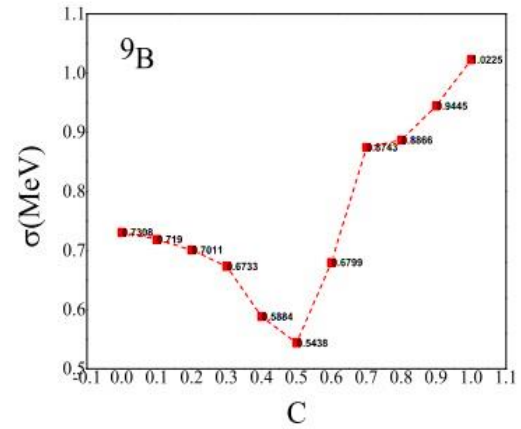


Fig.1. Energy levels as a function of the control parameter C for two-cluster nuclei.



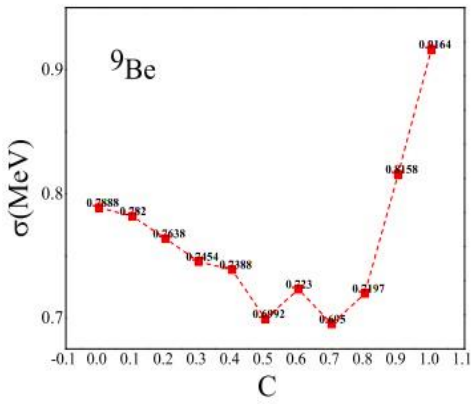


Fig.2. The root mean square deviation as a function of the control parameter C for two-cluster nuclei.

B. Expectation values of boson number

The other quantal order parameters that we mention here are the expectation values of the boson number operators. The expectation values of n_b are the significant objectives of phase transition. So, we calculated these values to show the treatment of phase transition. In order to calculate the expectation values of the b-boson number operator, we have to select the suitable roots. Given the proper amount of roots, we have calculated $< n_b >$ for two, three and four - clusters in even - even and odd-A nuclei.

Fig.3 shows the expectation values of the b-boson number operator for the lowest states even-even (left panel) and odd-A nuclei (right panel) as a function of control parameter for $N = 10$ bosons. The sudden change in these quantities shows the phase transition. Figures show that the expectation values of the number of vector-bosons remain approximately constant for a limit and only begin to change rapidly for the other limit. It can be seen from Fig.3 that due to the presence of the fermion, the transition is made sharper for even-even nuclei while it is made smoother for odd-A nuclei. We also found that the position of the critical point has been shifted by the addition of the odd particle with respect to the even case. As an outcome, the behavior of the odd and even systems at the corresponding critical points is rather similar.

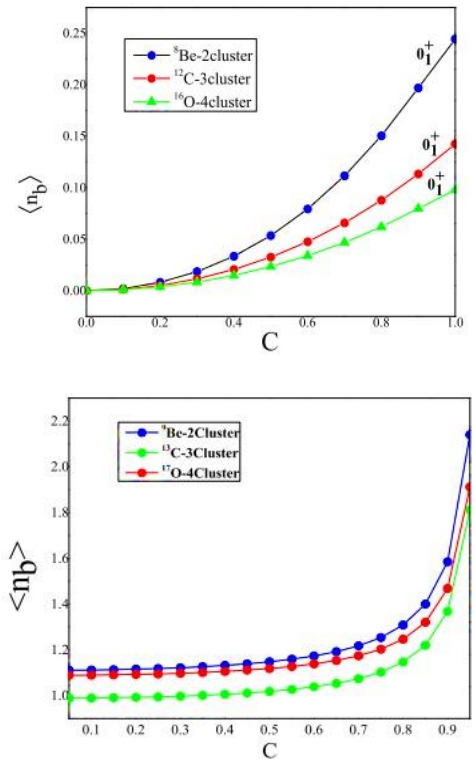


Fig.3. The expectation values of the vector-boson number operator for the lowest states as a function of control parameter C for N=10.

C. Calculated variation behavior of the overlap of the ground-state wave function

It has been shown previously that the overlap of the ground-state wave function with that in the dynamical symmetries may also serve as a signature of the phase transition [44-46]. We have calculated the overlap of the ground-state wave functions of the Hamiltonians (17) and (26) in $|< g.s.C_1|g.s.C_2 >|$ with $C_2 = 1$ for ${}^{9}_{4}Be$ and ${}^{10}_{5}B$. The obtained results are illustrated in Fig. 4. It indicates that the largest absolute value of the derivative of $|< g.s.C_1|g.s.C_2 >|$ with respect to C_1 occurs around the critical point $C_1 = 0.6$ for ${}^{9}_{4}Be$ and $C_1 = 0.4$ for ${}^{10}_{5}B$.

V. Conclusion

In this paper, we have studied the phase transitions of the algebraic cluster models. The ${}^{9}Be$, ${}^{9}B$ and ${}^{10}B$ nuclei were studied in the $SU(3) \leftrightarrow SO(4)$ phase transitions, related to the description of the relative motion of the cluster configurations. A solvable extended transitional Hamiltonian which is based on $SU(1,1)$ algebra is proposed to pave the way for a quantum phase transition between the spherical and the deformed phases. The validity of the presented parameters in the cluster-IBM and cluster-IBFM formulations has been investigated and it is seen that there exists a satisfactory agreement between the presented results and the experimental counterparts. We have presented here an analysis of quantum phase transitions in a system of N bosons and one fermion and shown that the addition of a fermion greatly modifies the critical value at which the phase transition occurs.

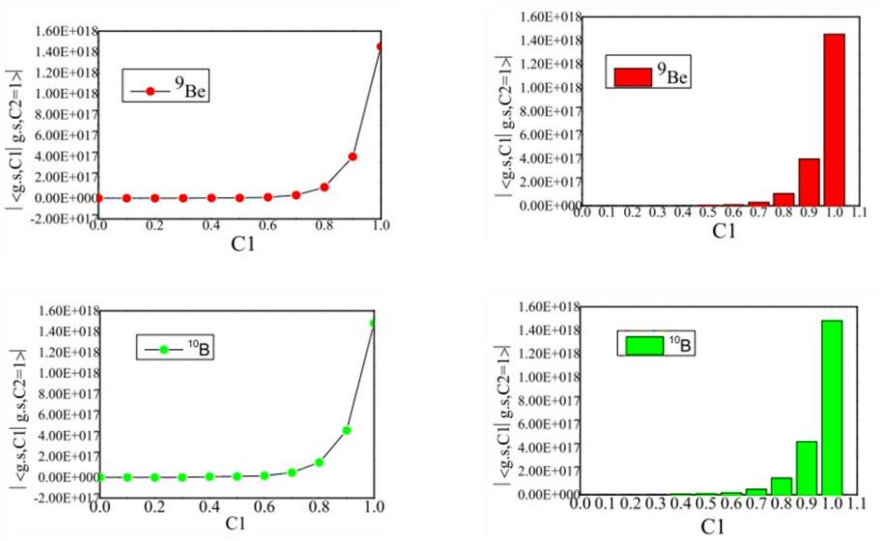


Fig. 4. The Calculated variation behavior of the overlap of the ground-state wave function as a function of control parameter C for N=10.

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