



The Λ -core Model for Studying the Binding Energies of Λ -hypernuclei under Pseudo-Spin Symmetry using the Hellmann Potential

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ABSTRACT

We have studied the binding energies of a group of single Λ -hypernuclei in a relativistic approach and modeled the single Λ -hypernuclei as a Λ -core binary system. Since the Hellmann potential is ideal for defining nucleon-core interaction and the single Λ -hypernuclei can be assumed as a single nucleon coupled to the whole nuclei but does not suffer from Pauli blocking, we have selected this potential for interaction between the Λ particle and the core. The time-independent Dirac equation is a reasonable option for defining the relativistic bound states corresponding to a spin-1/2 Λ hyperon in the hypernuclei. We solved this equation by using the Hellmann potential under the presence of pseudo-spin symmetry in terms of the generalized parametric Nikiforov-Uvarov method, a way to make the application of the Nikiforov-Uvarov method as plain as possible. Our results were in good agreement with experimental values and other theoretical works. Hence, this model applies to the Λ -hypernuclei.

Keywords: Hypernuclei; Λ -Hyperon; Dirac equation; Pseudo-spin symmetry; Nikiforov-Uvarov method.

I. Introductions

The Λ -nucleus interaction is the principal purpose of hypernuclear research [1]. Because of a new degree of freedom, the strangeness, the Λ hyperon does not suffer from Pauli blocking by nucleons of an ordinary nucleus [2, 3]. Hence, the Λ particle can penetrate the nucleus and form strongly bound hypernuclear states, the so-called single Λ -hypernuclei [3].

In this work, we model the single Λ -hypernuclei as a Λ -core binary system like a single nucleon coupled to the whole nuclei to calculate the binding energies of the Λ hyperon. We use the Hellmann potential [4] as the potential between the Λ particle

and the core. This potential is good enough for defining nucleon-core interaction [5]. Hence, this is suitable for the Λ -core system. Further, it is common to assume the single Λ -hypernuclei as a Λ -core binary system. The binding energies of the ground [6, 7] and first excited [7] states of various single Λ -hypernuclei recently were calculated in the non-relativistic approach by solving the Schrödinger equation for the Λ -core system with the Hulthén potential [8]. In addition, the binding energies of the ground and excited states and the root mean square (RMS) radii of several single Λ -hypernuclei were estimated in relativistic form

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using Woods–Saxon potential [9] under spin symmetry by the hyperon-core model [10]. Therefore, after calculating the binding energy of the Λ hyperon, we compare them with experimental values and the results of other theoretical works like ref. [10].

The rest of this paper adheres to the following sections:

- The first section refers to our analytical method to solve the Dirac equation, the parametric Nikiforov-Uvarov (PNU method).
- The second one refers to the solution of the Dirac equation with the suggested potential.
- The third one refers to the calculated binding energies of the ground and excited bound states of Λ .
- The last one is the conclusions.

II. The Review of Parametric Nikiforov–Uvarov Method

The NU method [11] is a powerful alternative to solve the Schrödinger, Dirac, and Klein–Gordon wave equations for a given kind of potential [12, 13]. This method is based on the second-order differential equations of the hypergeometric type by applying the special orthogonal functions [14, 15]. The basic reduced differential equation to start the NU method is the following equation [12, 14, 15]:

$$\Psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \Psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \Psi(s) = 0 \quad (1)$$

where $\Psi(s)$ is a function of the hypergeometric type, $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial.

The PNU method is a shortcut to the NU method, which has been used to solve Schrödinger [16], Dirac [17], and Klein-Gordon [18] wave equations. It is a way to make the application of the NU method easier without any requirement to check

the validity of the solution [12]. To solve a second-order equation by the PNU method, one should rewrite the equation in the below standard form for any potential [19]:

$$\left[\frac{d^2}{ds^2} + \frac{(c_1 - c_2s)}{s(1 - c_3s)} \frac{d}{ds} + \frac{-\varepsilon_1s^2 + \varepsilon_2s - \varepsilon_3}{s^2(1 - c_3s)^2} \right] R(s) = 0. \quad (2)$$

Based on the Nikiforov-Uvarov method, the energy equation and the corresponding wave functions become the following equations, respectively [14, 19]:

$$c_{2n} - (2n + 1)c_5 + (2n + 1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n - 1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0, \quad (3)$$

and

$$R(s) = s^{c_{12}}(1 - c_3s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n \left(c_{10} - 1, \frac{c_{11}}{c_3} - c_{10} - 1 \right) (1 - 2c_3s) \quad (4)$$

where $P_n^{(\mu, \nu)}(x)$ are Jacobi polynomials. Also, the c parameters are given as follows [14, 19]:

$$\begin{aligned} c_4 &= \frac{1}{2}(1 - c_1), & c_5 &= \frac{1}{2}(c_2 - 2c_3), \\ c_6 &= c_5^2 + \varepsilon_1, & c_7 &= 2c_4c_5 - \varepsilon_2, \\ c_8 &= c_4^2 + \varepsilon_3, & c_9 &= c_3c_7 + c_3^2c_8 + c_6, \\ c_{10} &= c_1 + 2c_4 + 2\sqrt{c_8}, \\ c_{11} &= c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}), \\ c_{12} &= c_4 + \sqrt{c_8}, \\ c_{13} &= c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}). \end{aligned} \quad (5)$$

In a special case when $c_3=0$, $R(s)$ becomes [14, 19]:

$$R(s) = s^{c_{12}} e^{c_{12}s} L_n^{c_{10}-1}(c_{11}s) \quad (6)$$

III. The Solution of the Dirac Equation with the Suggested Potential

The discussion of relativistic effects is consistently effective in some quantum mechanical cases [5]. The Dirac equation generally demonstrates the behavior of a spin-1/2 particle in relativistic quantum mechanics [13]. To characterize relativistic bound states of a spin-1/2 hyperon in the hypernuclei, we consider the wave function of the single hyperon satisfies the time-independent Dirac equation in the following form [19, 20]:

$$\left[\hat{\alpha} \cdot c \hat{\mathbf{p}} + \hat{\beta} (Mc^2 + S(r)) \right] \psi_{n_r, \kappa}(r) = [E - V(r)] \psi_{n_r, \kappa}(r), \quad (7)$$

where M , E , and $\hat{\mathbf{p}}$ are the single-particle rest mass, total relativistic energy, and the momentum operator, respectively. Furthermore, $S(r)$ and $V(r)$ represent the attractive scalar and repulsive vector potential. The momentum operator and the Dirac matrices are defined as follows:

$$\hat{\mathbf{p}} = -i\hbar\nabla, \quad \hat{\alpha} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (8)$$

where σ_i are two-dimensional Pauli matrices. Further, I is the identity matrix. The Dirac Hamiltonian in a central potential can commute with the total angular momentum \hat{j} and the spin-orbit coupling operator $\hat{\mathbf{K}} = \beta(\sigma \cdot \mathbf{L} + 1)$, where \mathbf{L} is the orbital angular momentum. The eigenvalue of the total angular momentum \hat{j} is j . The eigenvalues of the spin-orbit coupling operator $\hat{\mathbf{K}}$ are $\kappa = \pm(j+1/2)$. The positive sign is for the unaligned spin, and the negative sign is for the aligned spin. Thus, we can write the Dirac spinors in terms of the total angular momentum j , the spin-orbit quantum number κ , and the radial quantum number n_r in a central field as

$$\psi_{n_r, \kappa}(r, \theta, \phi) = \frac{1}{r} \begin{pmatrix} F_{n_r, \kappa}(r) Y_{jm}^l(\theta, \phi) \\ i G_{n_r, \kappa}(r) Y_{jm}^{\tilde{l}}(\theta, \phi) \end{pmatrix}, \quad (9)$$

where $F_{n_r, \kappa}(r)$ are the upper components, $G_{n_r, \kappa}(r)$ are the lower components of the Dirac spinors. $Y_{jm}^l(\theta, \phi)$ and $Y_{jm}^{\tilde{l}}(\theta, \phi)$ are the spherical harmonic functions, in which m is the projection of the angular momentum on the z-axis. In addition, l and \tilde{l} are the orbital angular momentum quantum numbers indicating the spin and pseudo-spin quantum numbers.

Previously, pseudo-spin symmetry was applied to study deformed nuclei and the nuclear super deformed states [17, 21, 22]. The pseudo-spin symmetry comes off by the near equality of the magnitude of $S(r)$ and $V(r)$, i.e., $S(r) = -V(r) + C_{ps}$ in hypernuclei where C_{ps} is a constant [23]. Therefore, to find the energy equation under the pseudo-spin symmetry, we derived the following second-order equation from Eq. (7)

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{1}{\hbar^2 c^2} (Mc^2 + E - \Delta(r))(Mc^2 - E) \right] G_{n_r, \kappa}(r) = 0 \quad (10)$$

by considering [17, 19]:

$$\begin{aligned} \Sigma(r) &= V(r) + S(r) = 0 \\ \Delta(r) &= V(r) - S(r) \\ S(r) &\approx -V(r) \end{aligned} \quad (11)$$

The Hellmann potential [4] is defined as [17]:

$$V(r) = -\frac{a}{r} + \frac{b}{r} e^{-\alpha r} \quad (12)$$

where a and b as the Coulomb and the Yukawa potentials and α as the screening parameter.

We consider the following transformations

$$\tilde{\beta} = \frac{[E^2 - M^2 c^4]}{\hbar^2 c^2}, \quad \tilde{\gamma} = \frac{[Mc^2 - E]}{\hbar^2 c^2} \quad (13)$$

and rewrite Eq. (10) as

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \tilde{\beta} + \tilde{\gamma}\Delta(r) \right] G_{nr,\kappa}(r) = 0 \quad (14)$$

Since Eq. (14) is a combination of the exponential and inverse square potentials, it is not exactly solvable, in general [19]. Hence, we replace the centrifugal term with the improved Pekeris approximation [5, 17, 24].

$$\frac{1}{r^2} \approx \frac{\alpha^2}{(1 - e^{-\alpha r})^2} \quad \alpha \leq 1 \quad (15)$$

Using Eq. (15), $s = e^{-\alpha r}$, and $G_{nr,\kappa}(r) \equiv R(s)$, Eq (8) becomes the below form:

$$\left[\frac{d^2}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{d}{ds} + \frac{-\varepsilon_1 s^2 + \varepsilon_2 s - \varepsilon_3}{s^2(1-s)^2} \right] R(s) = 0, \quad (16)$$

where

$$\begin{aligned} \varepsilon_1 &= -\frac{\tilde{\beta}}{\alpha^2} + \frac{2b}{\alpha} \tilde{\gamma} \\ \varepsilon_2 &= -\frac{2\tilde{\beta}}{\alpha^2} + \frac{2a}{\alpha} \tilde{\gamma} + \frac{2b}{\alpha} \tilde{\gamma} \\ \varepsilon_3 &= -\frac{\tilde{\beta}}{\alpha^2} + \frac{2a}{\alpha} \tilde{\gamma} + \kappa(\kappa-1) \end{aligned} \quad (17)$$

Therefore, by using the PNU method and obtaining the following parameters,

$$\begin{aligned} c_1 = c_2 = c_3 = 1; \quad c_4 = 0; \quad c_5 = -\frac{1}{2}; \\ c_6 = \frac{1}{4} + \varepsilon_1; \quad c_7 = -\varepsilon_2; \quad c_8 = \varepsilon_3; \\ c_9 = \frac{1}{4} + \varepsilon_1 - \varepsilon_2 + \varepsilon_3; \quad c_{10} = 1 + 2\sqrt{\varepsilon_3}; \\ c_{11} = 2 + 2\left(\sqrt{\frac{1}{4} + \varepsilon_1 - \varepsilon_2 + \varepsilon_3} + \sqrt{\varepsilon_3}\right); \quad c_{12} = \sqrt{\varepsilon_3}; \\ c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} + \varepsilon_1 - \varepsilon_2 + \varepsilon_3} + \sqrt{\varepsilon_3}\right). \end{aligned} \quad (18)$$

the energy equation and $R(s)$ under the pseudo-spin symmetry, respectively, can be written as:

$$\begin{aligned} n^2 + n + \frac{1}{2} \\ + (2n+1) \left[\sqrt{\frac{1}{4} + \kappa(\kappa-1)} + \sqrt{-\frac{\tilde{\beta}}{\alpha^2} + \frac{2a}{\alpha} \tilde{\gamma} + \kappa(\kappa-1)} \right] \\ - \frac{2\tilde{\gamma}}{\alpha} (b-a) + 2\kappa(\kappa-1) \\ + 2\sqrt{\left(\frac{1}{4} + \kappa(\kappa-1)\right) \left(-\frac{\tilde{\beta}}{\alpha^2} + \frac{2a}{\alpha} \tilde{\gamma} + \kappa(\kappa-1)\right)} = 0 \end{aligned} \quad (19)$$

and

$$\begin{aligned} R(s) &= s^{c_{12}} (1-s)^{-c_{12} - c_{13}} \\ P_n^{(c_{10}-1, c_{11}-c_{10}-1)}(1-2s) \end{aligned} \quad (20)$$

IV. Calculation Binding Energies of the Ground and Excited Bound States of Λ

Our objective is to find the binding energies of the ground and excited bound states of Λ particle by using the relativistic energy equation. Therefore, we apply: $Mc^2 = 1115$ MeV [10]; $E = -(-E_\Lambda + Mc^2)$ [10], where E_Λ is the binding energy of the Λ hyperon of the particular state; $n=1$ [17]; and $\kappa = -\tilde{l}$ [17] to find strengths of Coulomb (a) and the Yukawa (b) potentials in MeV and α as the screening parameter in fm^{-1} by fitting with the experimental results. At last, we calculate the binding energies of the 1s and 1p states of the Λ particle in a group of single Λ -hypernuclei and list them in tables 1 and 2. Besides, we calculate the binding energies of Λ in d and f states for ${}_\Lambda^{208}\text{Pb}$, as seen in table 3. We apply $n-1=0$ and $\kappa = \tilde{l} = 3$ [17] to calculate the f state for ${}_\Lambda^{208}\text{Pb}$ because these amounts give better agreement with the experimental value of the binding energy of Λ in this state.

V. Conclusions

In this work, we found the energy equation under pseudo-spin symmetry. Then, we calculated the optimum form of Hellmann potential for any of the studied Λ -hypernuclei and the binding energy of Λ for them. Our results agree with the experimental values and other theoretical works like ref. [10]. Hence, this model applies to the Λ -hypernuclei.

Table 1. The ground state binding energy of Λ (MeV).

Hypernuclei	Coefficients of the potential		E_{Λ} -Our	E_{Λ} -Exp. [25]	E_{Λ} -Other. [10]	E_{Λ} -Other. [7]
$\Lambda^{13}\text{C}$	α	0.15				
	a	0.4096	11.9197	11.69	11.41	11.447
	b	-58.6925				
$\Lambda^{16}\text{N}$	α	0.12				
	a	1.85156	12.5293	13.67	11.40	12.967
	b	-57.8448				
$\Lambda^{16}\text{O}$	α	0.12				
	a	2.1925	13.2626	13.0	12.86	12.645
	b	-56.5075				
$\Lambda^{28}\text{Si}$	α	0.05				
	a	4.3675	17.5490	17.20	17.09	17.240
	b	-53.4810				
$\Lambda^{32}\text{S}$	α	0.04				
	a	3.3773	17.8397	17.50	17.33	-
	b	-53.6510				
$\Lambda^{40}\text{Ca}$	α	0.083				
	a	-45.0112	18.9076	18.70±1.1 [26]	18.39	18.493
	b	-95.5675				
$\Lambda^{51}\text{V}$	α	0.084				
	a	-56.5173	21.8460	21.50	21.65	18.964
	b	-108.7796				
$\Lambda^{208}\text{Pb}$	α	0.025				
	a	-388.5	27.0820	26.90	26.83	20.624
	b	-425.4				

Table 2. The first excited state binding energy of Λ (MeV).

Hypernuclei	E_{Λ} -Our	E_{Λ} -Exp. [25]	E_{Λ} -Other. [10]	E_{Λ} -Other. [7]
$\Lambda^{13}\text{C}$	0.8083	0.8	0.85	2.24
$\Lambda^{16}\text{N}$	2.2686	2.84	2.79	2.53
$\Lambda^{16}\text{O}$	2.5287	2.5	2.32	-
$\Lambda^{28}\text{Si}$	7.6734	7.6	7.68	8.543
$\Lambda^{32}\text{S}$	8.3207	8.20	8.25	-
$\Lambda^{40}\text{Ca}$	11.0832	11.00	11.06	8.138
$\Lambda^{51}\text{V}$	13.4855	13.4	13.48	9.09
$\Lambda^{208}\text{Pb}$	23.0976	22.50	22.69	14.429

Table 3. d and f binding energies of Λ in $\Lambda^{208}\text{Pb}$ (MeV).

$\Lambda^{208}\text{Pb}$		E_{Λ} -Our	E_{Λ} -Exp. [25]	E_{Λ} -Other. [10]
	d	17.3687	17.4	15.25
f	12.1182	12.3	12.10	

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