# Investigation of Structure and Nuclear Shape Phase Transition in Odd Nuclei in a multi-j model

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## ABSTRACT

In this paper, we study the nature of the dynamics in second-order Quantum Phase Transition (QPT) between vibrational ($U^{BF}(5)$) and $\gamma$-unstable ($O^{BF}(6)$) nuclear shapes. Using a transitional Hamiltonian according to an affine SU(1,1) algebra in combination with a coherent state formalism, Shape Phase Transitions (SPT) in odd-nuclei in the framework of the Interacting Boson Fermion Model (IBFM) are investigated. Classical analysis reveals a change in the system along with the transition in a critical point. The role of a fermion with angular momentum j at the critical point on quantum phase transitions in bosonic systems is investigated via a semi-classical approach. The effect of the coupling of the odd particle to an even-even boson core is discussed along with the shape transition and, in particular, at the critical point. Our study confirms the importance of the odd nuclei as necessary signatures to characterize the occurrence of the phase transition and determine the critical point's precise position.

*Keywords*: Affine SU(1, 1) algebra, -Unstable nuclear shapes, Quantum Phase Transition, Interacting Boson Fermion Model

## I. Introductions

When the number of nucleons modifies, the quantum shape phase transition happens as the system's structure changes from one character to another. The existence of shape phase transitional behavior manifests in both even-even and odd-A nuclei. Phase transitions studies have mostly been done on even-even and odd mass systems within the interacting boson model (IBM)1 and the interacting boson fermion model (IBFM)2.

The interacting boson model (IBM) has three dynamical symmetries signified by U(5), SU(3), and O(6), which are usually associated with spherical (U(5)), axially deformed (SU(3)) and -unstable (O(6)) shapes. A first-order QPT is observed between U(5)-SU(3), while the transition from U(5) to O(6) is second -order (continuous)

[1-3]. Quantum phase transitions in odd-A nuclei as a system of bosons and a single fermion were studied both classically and quantum mechanically in Ref.[4] It was displayed that the existence of the odd fermion powerfully affects the position and character of the phase transition, particularly at the critical point in which the phase transition happens. It is found that the existence of a single fermion changes the critical value at which the transition occurs for any amount of the coupling constant [5]. Several explicit studies by Alonso et al. [6-10] and Boyukata et al. [11] were done to study the quantum phase transitions in boson-fermion systems. They proposed an analytically solvable model E(5/12) to explain odd nuclei at the critical point in the transition from the vibrational to gamma-unstable phase. In Ref. [12], the role of a

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single fermion on the shape phase transition was also studied. It was shown that unpaired particle describes the features at the critical points in atomic nuclei. They said that the outstanding features of shape phase transitions in odd-A nuclei don't change by coupling the single fermion to the even-even case. The effect of the coupling of the odd particle in the during of shape transition from U(5) to SU(3) and the transition from U(5) to O(6) in odd-A nuclei, in particular, at the critical point was discussed in Ref. [13]. The ground-state energy surface was obtained in terms of the shape variables (and) for the even core, and odd-even energy surfaces. It was found that the core-fermion coupling gives rise to a smoother transition than in the even-core case. D. Bucurescu and N. V. Zamfir studied quantum shape phase transition in odd-A nuclei. The evolution of level structures in odd-mass nuclei and their even-even core nuclei is studied by correlations between ratios of excitation energies in both odd-mass nuclei.

Xiang-Ru Yu et al. [14] investigated the effects of coupling of a single particle on shape phase transitions and phase coexistence in odd-even nuclei in the framework of the interacting boson-fermion model. Their results displayed that a single particle may impress different types of SPT in various ways. They have shown that phase coexistence can appear in the critical region and thus be taken as a sign of the shape phase transitions in odd-even nuclei.

In this paper, a systematic study of spherical to γ-unstable shape phase transition and the effects of coupling a single particle on SPTs in the framework of IBFM is present. Particularly, how the existence of an odd particle can influence SPT is investigated. To consider an even-even boson core that performs a transition from spherical to γ-unstable shapes with a single nucleon in orbits to describe Odd-A nuclei. The E(5|12) model of that coupling of an even core that experiences a transition from spherical U(5) to γ-unstable O(6) situation to a particle moving in the 1/2, 3/2, and 5/2 orbitals is proposed by using the affine SU(1,1)

Lie algebra and the Bethe ansatz technique in the framework of the IBFM. Using the Catastrophe theory, the suggested Hamiltonians, and the coherent state of boson-fermion systems, we have created the corresponding energy surfaces in terms of the Hamiltonian parameters and the shape variables. The energy surfaces and equilibrium values along the shape phase transition are explored in the odd-A nuclei for different control parameters.

This paper is organized as follows: Section 2 briefly summarizes theoretical aspects of transitional Hamiltonian and affine SU(1,1) algebraic technique. Sections 3 include calculating energy surface and sects. 4 and 5 are devoted to the discussion, Classical Analysis, and Conclusions.

## II. The Applied Transitional Hamiltonians concerning the Affine SU(1, 1) Algebra

In this section, we study the coupling of an even core that undergoes a transition from spherical U(5) to $\gamma$-unstable O(6) situation to a particle moving in the 1/2, 3/2, and 5/2 orbitals by using the affine SU(1,1) Lie Algebra and the Bethe ansatz technique within an infinite-dimensional Lie algebra in the framework of the interacting boson-fermion model. We have used the same formalism to extend IBFM calculation to the case that a j (1/2, 3/2, and 5/2) fermion coupled to a boson core. As shown in Refs. [9-10], the E(5|12) model is constructed by considering the case of a collective core with the E(5) symmetry coupled with a single particle in a j = 1/2, 3/2, and 5/2 orbit. For better clarification, we show in Fig.1 the lattice of algebras relevant to our problem. By employing the generators of Algebra SU(1,1) and Casimir operators of subalgebras, the following Hamiltonian for the transitional region between $U^{BF}(5)$ - $O^{BF}(6)$ limits is prepared

$$(1)$$

$$H = g S_0^+ S_0^- + \alpha S_1^0 + \delta C_2\left(O^{BF}(5)\right) + \gamma C_2\left(O^{BF}(3)\right) + \eta C_2\left(spin^{BF}(3)\right)$$

where $C_2(O^{BF}(5))$, $C_2(O^{BF}(3))$ and $C_2(spin^{BF}(3)))$ are the quadratic Casimir operators of the $O^{BF}(5)$,

$O^{BF}(3)$ and $spin^{BF}(3)$ algebras, respectively. This choice of the fermion space is such that one can usefully visualize the three angular momenta as arising from the coupling of a pseudospin 1/2 with a pseudo-orbital angular momentum 0 or 2. One recovers, in the extreme cases, the Hamiltonian Eq. (1) associated with $O^{BF}(6) \otimes U_s^F(2)$ Hamiltonian when $c_s = c_d$ and with $U^{BF}(5) \otimes U_s^F(2)$ Hamiltonian if $c_s = 0$ and $c_d \neq 0$. Thus, As mentioned, because only the boson core experience the transition and fermion coupled to the boson core, the situation just corresponds $O^{BF}(6) \otimes U_s^F(2) \leftrightarrow U^{BF}(5) \otimes U_s^F(2)$ to the transitional region.
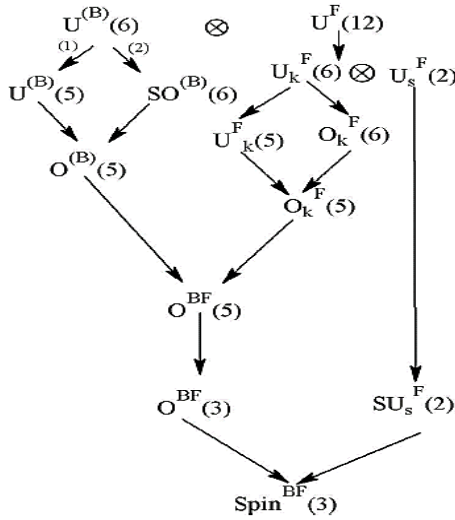


**Fig. 1** The lattice of algebras in the case that a system of $N_B$ bosons (with L=0, 2) coupled to a fermion with angular momentum j=1/2, 3/2, and 5/2.

For evaluating the eigenvalues of Hamiltonians Eq. (1), the eigenstates are considered as

$$\left| k; \nu_s \nu_d n_\Delta JM \right\rangle = \sum_{n_i \in Z} a_{n_1,n_2,...,n_k} X_1^{n_1} X_2^{n_2} \cdots X_k^{n_k} S_{n_1}^+ S_{n_2}^+ \cdots S_{n_k}^+ \left| lw \right\rangle^{BF} \tag{2}$$

Eigenstates of Hamiltonian Eq.(1) can obtain using the Fourier-Laurent expansion of eigenstates and SU(1,1) generators in terms of c-unknown number parameters $x_i$ with i=1,2,...,k. It means, one can consider the eigenstates as [19-20]

$$\left| k; \nu_s \nu_d n_\Delta JM \right\rangle = \Theta' S_{x_1}^+ S_{x_2}^+ \cdots S_{x_k}^+ \left| lw \right\rangle^{BF} \tag{3}$$

Using the CG coefficient, we can obtain the lowest weight state, $\left| lw \right\rangle^{BF}$, in terms of boson and fermion parts

$$\left| lw \right\rangle^{BF} = \sum_{j=\frac{1}{2},\frac{3}{2},\frac{5}{2}} \sum_{m_j=-j}^{j} C_{M,M_L,m_j}^{J,L,j} \left| lw \right\rangle_{M_L}^{B} \otimes \left| j,m \right\rangle \tag{4}$$

$$\tag{5}$$

$$\left| lw \right\rangle_{M_L}^{B} = \left| N_B, k_d = \frac{1}{2}(\nu_d + \frac{5}{2}), \mu_d = \frac{1}{2}(n_d + \frac{5}{2}), k_s = \frac{1}{2}(\nu_s + \frac{1}{2}), \mu_s = \frac{1}{2}(n_d + \frac{1}{2}), L, M_L \right\rangle$$

Therefore, our states are characterized by the quantum numbers $(\tau_1, \tau_2)$ from $SO^{BF}(5)$, L from $SO^{BF}(3)$ and J of $spin^{BF}(3)$. The eigenvalues of Hamiltonian Eq.(1) can then be expressed

$$\tag{6}$$

$$E^{(k)} = h^{(k)} + \alpha \Lambda_1^0 + \delta(\tau_1(\tau_1+3) + \tau_2(\tau_2+1)) + \gamma L(L+1) + \eta J(J+1)$$

**III. Calculating of Energy Surface**
We can investigate the classical limit of the considered model in the framework of a coherent state[4, 5, 8-10]. The cohesive state formalisms of IBM and IBFM [4, 5, 20, 25, 26] connect the algebraic and geometric descriptions of three dynamical symmetry limits and also allows the study of transitions among them. Using this formalism, one can evaluate the ground state energy as a function of shape variables $\beta$ and $\gamma$, i.e., deformation parameters, similar to what has been done for $U(5) \leftrightarrow SO(6)$ transition.

The ground state energy surface is obtained by calculating the expectation value of the boson Hamiltonian in the coherent state [25-26]

$$\left| N_B, \alpha \right\rangle = \left( s^\dagger + \sum_m \alpha_m d_m^\dagger \right)^{N_B} \left| 0 \right\rangle \tag{7}$$

Where $\left| 0 \right\rangle$ is the boson vacuum state, $s^\dagger$ and $d_m^\dagger$ are the boson operators of the IBM, and parameter

$\alpha_m$ can be related to the deformation collective parameters of the quadrupole

$$\alpha_0 = \beta\cos(\gamma), \quad \alpha_{\pm 1} = 0, \quad \alpha_{\pm 2} = \frac{\beta\sin(\gamma)}{\sqrt{2}}, \tag{8}$$

The $\beta$ variable measures the axial deviation from sphericity, and the angle variable $\gamma$ controls the departure from axial deformation [1, 25-26]. Energy surface would determine using

$$E(N_B,\alpha) = \frac{\langle N_B,\alpha \mid H \mid N_B,\alpha\rangle}{\langle N_B,\alpha \mid N_B,\alpha\rangle}, \tag{9}$$

Intrinsic states for the mixed boson-fermion system can be constructed by coupling the odd single-particle states to the coherent states of the even core [4, 5, 20]

$$\varphi = \mid N_B,\alpha\rangle \otimes \mid j,m_j\rangle, \tag{10}$$

The basis for the diagonalization of the fermion part is as

$$\mid j,m_j\rangle = a^\dagger_{j,m_j}\mid 0\rangle, \quad m_j = \pm j,\pm(j-1),...,\pm\frac{1}{2} \tag{11}$$

where $m_j$ is the projection of the total angular momentum j on the symmetry axis.

Intrinsic states for the hybrid boson-fermion system can be constructed by coupling the odd single-particle shapes to the coherent states of the even core [4, 5, 9, 10, 20]. So, the expectation value of $H_B$ in the boson condensate is given as

$$E = \frac{N_B(N_B-1)}{4(1+\beta^2)^2}\{c_s^2 + 2c_s c_d\beta^2 + c_d^2\beta^4\} + \frac{\alpha}{4}c_s^2\left(\frac{2N_B}{1+\beta^2}+1\right)$$
$$+ \frac{\alpha}{4}c_d^2\left(\frac{2\beta^2 N_B}{1+\beta^2}+5\right) + 2\delta\frac{\beta^2 N_B}{1+\beta^2} + 12\gamma\frac{\beta^2 N_B}{1+\beta^2} + 12\eta\frac{\beta^2 N_B}{1+\beta^2} \tag{12}$$

By integrating out the boson degrees of freedom, i.e.. by taking the expectation value of $H_F$ and $V_{BF}$ in the boson condensate, one can obtain the expectation value of the $H_F + V_{BF}$ in the boson condensate as

$$E = E_B + 4\delta\frac{N_B\beta^2\cos^2\gamma}{1+\beta^2}\langle 2020|30\rangle\left\{\left\langle K_0^{(3)}(2,2)\right\rangle + \left\langle K_2^{(3)}(2,2)\right\rangle\right\}$$
$$+ \left\langle K^{(1)}(2,2)\cdot K^{(1)}(2,2)\right\rangle \times (4\delta + 20\gamma + 20\eta) +$$
$$4\delta\left\langle K^{(3)}(2,2)\cdot K^{(3)}(2,2)\right\rangle + \eta\left\langle S^{(1)}\left(\frac{1}{2},\frac{1}{2}\right)\cdot S^{(1)}\left(\frac{1}{2},\frac{1}{2}\right)\right\rangle$$
$$- \sqrt{20}\eta\left\langle K^{(1)}(2,2)\cdot S^{(1)}\left(\frac{1}{2},\frac{1}{2}\right) + S^{(1)}\left(\frac{1}{2},\frac{1}{2}\right)\cdot K^{(1)}(2,2)\right\rangle \tag{13}$$

That

$$K_\mu^\lambda(k,k') = -\sum_s\sum_{j,j'}\sqrt{(2j+1)(2j'+1)}(-1)^{k+s+j'+\lambda}\begin{Bmatrix} j & j' & \lambda \\ k' & k & s \end{Bmatrix}A_\mu^{(\lambda)}(j,j')$$
$$S_\mu^\lambda(s,s') = -\sum_k\sum_{j,j'}\sqrt{(2j+1)(2j'+1)}(-1)^{k+s'+j+\lambda}\begin{Bmatrix} s & s' & \lambda \\ j' & j & k \end{Bmatrix}A_\mu^{(\lambda)}(j,j')$$
$$A_\mu^{(\lambda)}(j,j') = \left[a_j^\dagger \times \tilde{a}_{j'}\right]_\mu^{(\lambda)} \tag{14}$$

The curly bracket in Eqs.(14) is a 6j-symbol. The basis for the diagonalization of the fermion part is as Eq. (11). To calculate Eq. (13), we have used the following relations

$$\left\langle A_\mu^{(\lambda)}(j_1,j_2)\right\rangle = \left\langle j,m|\left(a_{j_1}^\dagger \times \tilde{a}_{j_2}\right)_\mu^\lambda|j,m\right\rangle$$
$$= \sum_{j_1,j_2}\sum_{m_1,m_2}(-1)^{j_2-m_2}\langle j_1 m_1 j_2 m_2|\lambda\mu\rangle\delta_{j,j_1}\delta_{j,j_2}\delta_{m,m_1}\delta_{m,-m_2} \tag{15}$$

$$\langle j,m|A_\mu^{(\lambda)}(j_1,j_2)A_{-\mu}^{(\lambda)}(j_3,j_4)|j,m\rangle =$$
$$\sum_{m_1,m_2}\sum_{m_3,m_4}(-1)^{j_2+j_4-m_4-m_2}\langle j_1 m_1 j_2 m_2|\lambda\mu\rangle$$
$$\times\langle j_3 m_3 j_4 m_4|\lambda-\mu\rangle\delta_{j,j_4}\delta_{j,j_1}\delta_{j_2,j_3}\delta_{m_3,m_2}\delta_{m,-m_4}\delta_{m,m_1} \tag{16}$$

Thus, if $j = j_1 = j_2$, $m_1 = -m_2$, $\mu = 0$ and then $\left\langle A_\mu^\lambda\right\rangle \neq 0$.

## IV. Discussion and Classical Analysis
### A. Energy surface
The evolution of the energy surfaces along with the shape phase transition for the boson core and the odd-even systems considering different angular momenta j=1/2, 3/2, and 5/2, orbits are displayed in Figures 2 and 3, respectively. We considered a number of bosons $N_B = 10$ for the boson core in all cases.
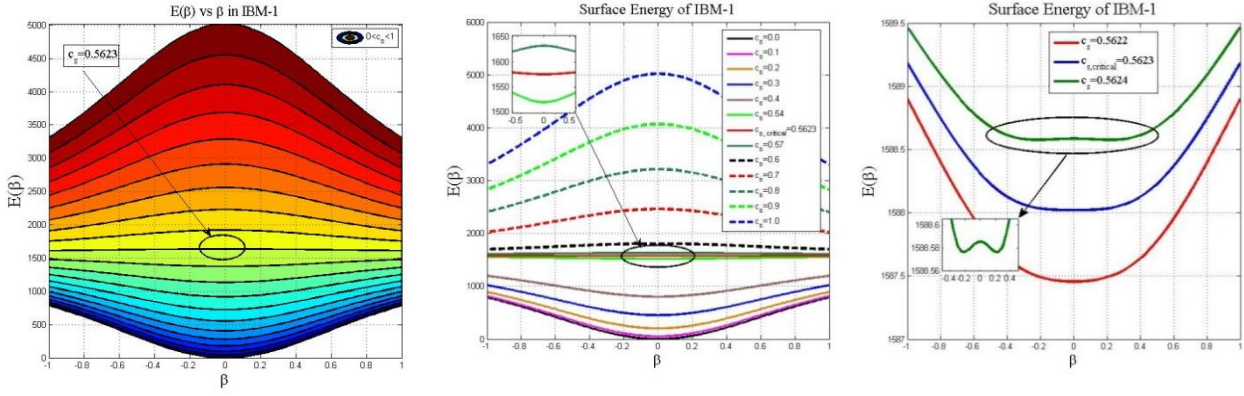
**Fig.2** The energy surfaces for an even-even case, as a function of the deformation parameter $\beta$ for different values of the control parameter $c_s$
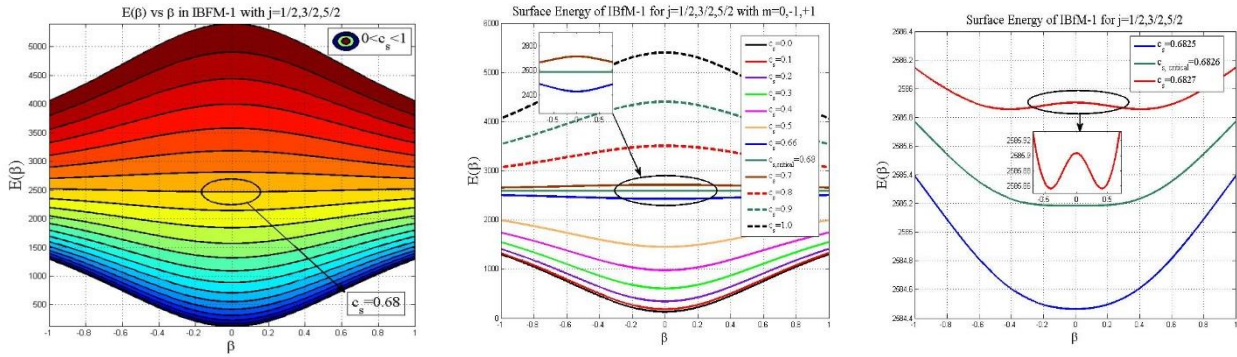


**Fig.3** The energy surfaces for an Odd-A nuclei with j=1/2, 3/2, and 5/2 as a function of the deformation parameter $\beta$ for different control parameter values $c_s$.

All energy surfaces have been calculated within a coherent state approach. To obtain energy surfaces, we need to specify Hamiltonian parameters Eq.(1). The Hamiltonian parameters, namely $\alpha$, $\eta$, $\delta$ and $\gamma$, used in the present work are shown in Table 1.

**Table 1.** Parameters of Hamiltonians used to calculate the surface energy. All parameters are given in keV.

| Nucleus | g | $\alpha$ | $\delta$ | $\gamma$ | $\eta$ |
|---|---|---|---|---|---|
| IBM[15] | 1 | 1000 | -95 | -143 | - |
| IBFM with j=1/2,3/2,5/2 (Eq.16) | 1 | 1000 | -95 | -143 | 136.5 |

In our calculation, we take $c_d\,(=1)$ constant values and $c_s$ changes between 0 and 1. By using Eq.(2.19) in Ref.[15], we first show in Fig.2 the energy surfaces for an even-even case as a function of the deformation parameter $\beta$ for different

values of the control parameter $c_s$. The other control parameter values $c_s$ in Figs are demonstrated by different lines. For $c_s = 0$, the lowest line, the even-even nucleus is spherical and it changes to well defined $\gamma$-unstable deformed shape for $c_s = 1$ (uppermost line). At the critical point, $c_{s,critical} = 0.5623$, the energy surface presents a very flat minimum as happens for the E(5) critical point model. One can notice that the surface energy is instead very flat where the phase transition occurs that corresponds to the essential point of the system. Then, we have studied the same transition for a boson-fermion case but considering different angular momenta j.

We calculated energy surfaces for boson-fermion systems that perform a transition from spherical to

$\gamma$ -unstable shapes with a single nucleon in j = 1/2, j = 3/2 and 5/2 orbits to describe Odd-A nuclei. The behavior of the energy surface as a function of the control parameter $c_s$ for the different j cases is depicted in Figs.(3) and (4), respectively. In analysis, it can be said that, the evolution of the energy surfaces suggests the SPT critical point $c_s = 0.68$ for j = 1/2, j = 3/2 and 5/2. The nature of the crucial point is changed in these control parameters. We see from the figures that the odd particle drives the system toward an unstable shapein these cases. This gives rise to an effective shift of the critical point. For example, for the odd-A systems, the critical issues move to $c_{s,critical} = 0.68$ (for j=1/2,3/2 5/2 orbits), where the corresponding energy surface becomes very flat.

Also, we show in Fig.(4), that the energy surfaces in the vibrational cases ( $c_s = 0$, leading to the $U^{BF}(5)$ dynamical symmetry) and $\gamma$ -unstable cases ( $c_s = 1$, leading to the $SO^{BF}(6)$ dynamical symmetry) as a function of the deformation

parameter $\beta$. As shown in Fig. 4, the energy curves for all j values in the spherical and $\gamma$ -unstable cases present the same features like that for the even-even core.

In these vibrational and $\gamma$ -unstable cases, the of the extra particle is not changing the system's features. But the situation is different around the critical point. While the results at the critical end indicate clearly the extra particle drives the system toward $\gamma$ -unstable shape.

To display how the energy surfaces change as a function of the control parameter $c_s$ and $-1 \leq \beta \leq 1$ are shown in Fig.(5). The Figs show how the energy levels as a function of the control parameter $c_s$ evolve from one dynamical symmetry limit to the other. It can be seen from Fig.(5) that numerous level crossings occur, especially in the region around $c_{s,critical}$. The crossings are since $V_d$, O(5) quantum number called seniority is preserved along the whole path between $\gamma$ -unstable and vibrational shapes.
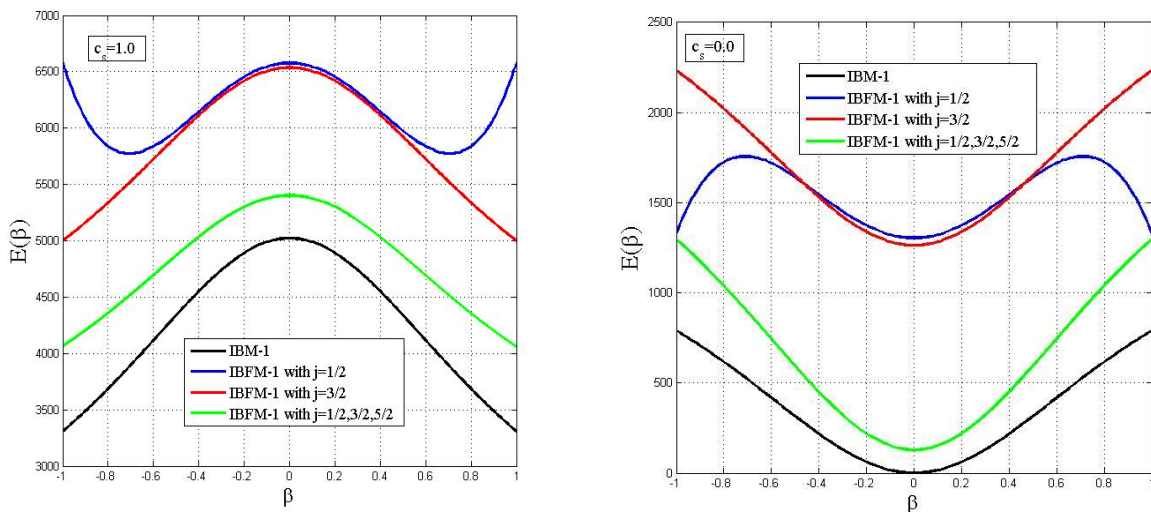


**Fig.4** The energy surfaces as a function of the deformation parameter $\beta$ for vibrational and $\gamma$ -unstable deformed cases.
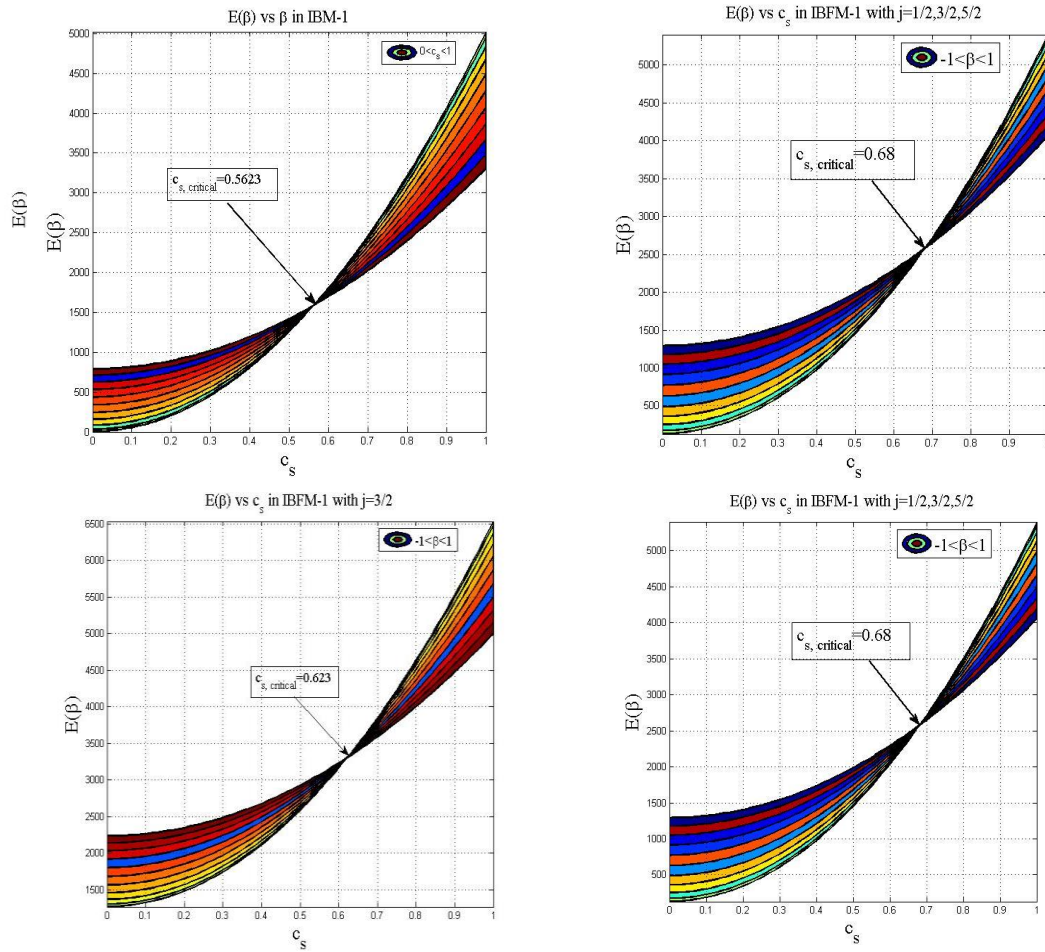
**Fig.5** The energy surfaces as a function of the control parameter $c_s$ for even-even and Odd-A nuclei.

To display how the energy surfaces change within the whole range of the $c_s$ and $\beta$ parameters, the energy surfaces $E(c_s, \beta)$ calculated. Once the eigenvalues have been obtained, we can display how the energy levels change within the entire range of the $c_s$ and $\beta$ parameters. Fig.(6) shows the ground energy surfaces for the even-even and odd-A nuclei. The calculations are performed by considering the same fit parameters reported in Table.(1). Fig. (6) shows how the energy levels function as the control parameter $c_s$ and $\beta$ evolve from one dynamical symmetry limit to the other.
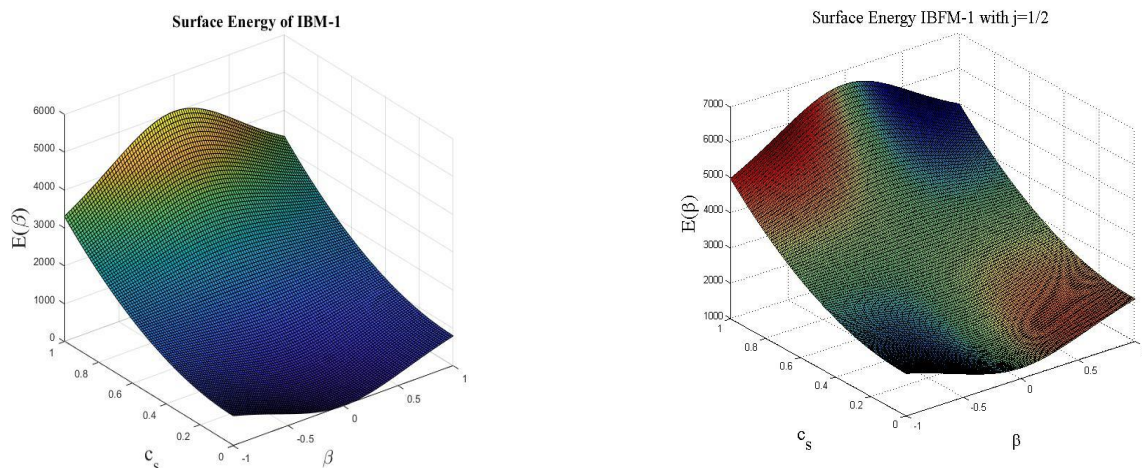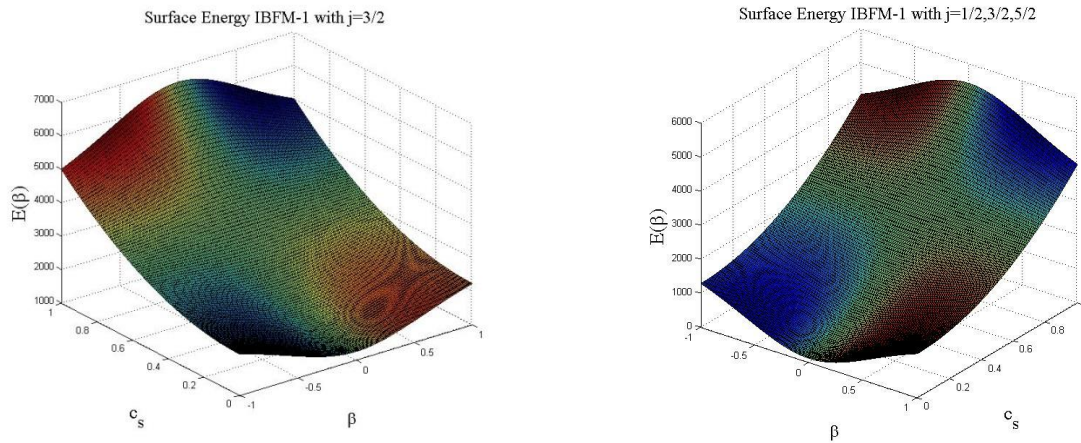
**Fig.6** The surface energy for odd-A nuclei as a function of $c_s$ and $\beta$ control parameters.
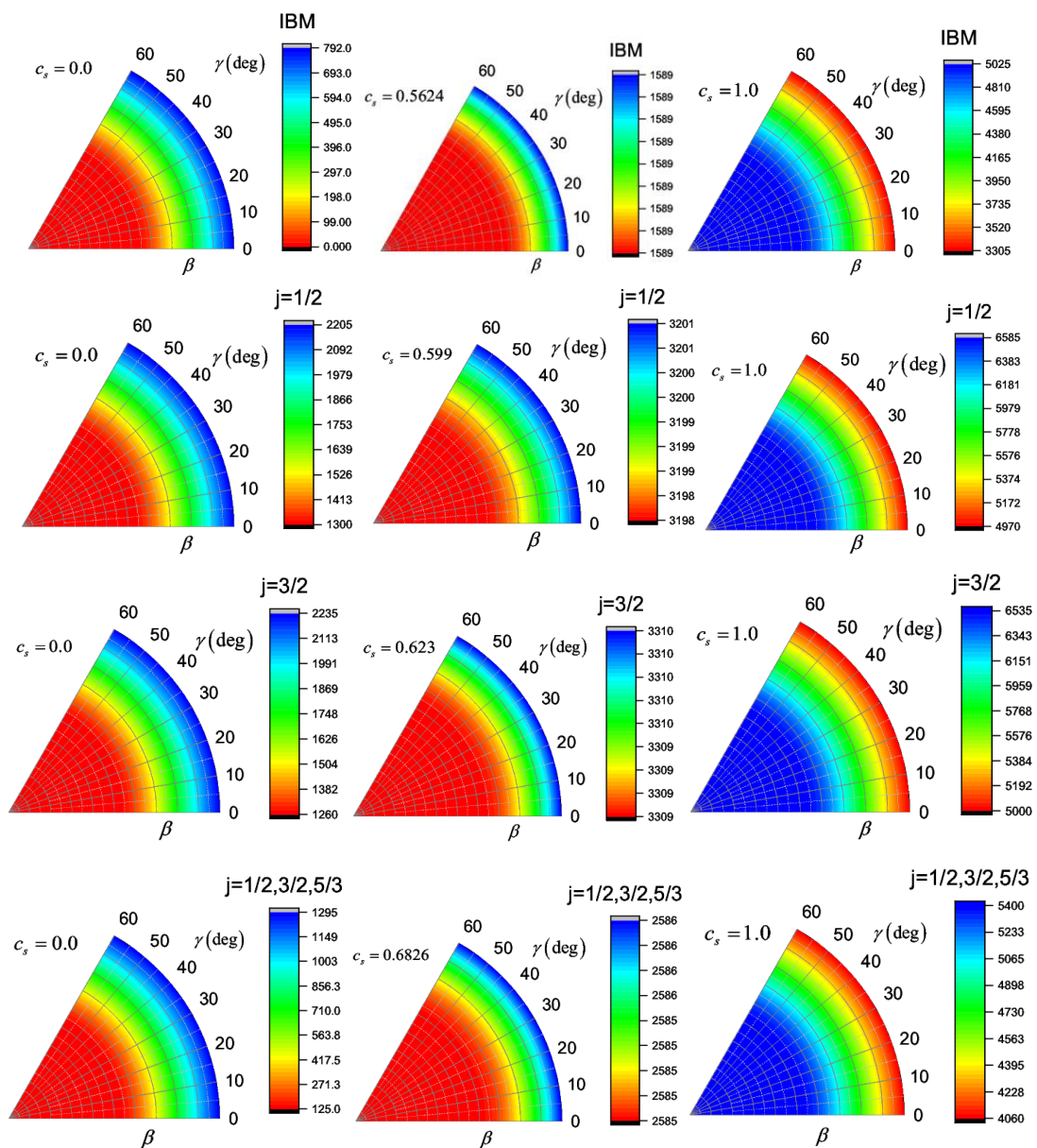


**Fig.7** Even-even and odd-A energy surfaces. The surface energy of odd-A nuclei in the $\beta$-$\gamma$ plane are shown for different values $c_s$.

**V. Equilibrium values: classical order parameter**

By minimization of $E(\beta,\gamma)$ concerning $\beta$ for any control parameter, the optimal values $\beta_e$ are obtained[4,5,11,14]. The $\beta_e$ is a suitable classical order parameter to determine the order and type of the SPT. For a first-order SPT, $\beta_e$ is not continuous as a function of the control parameter. While for a second-order SPT, $\beta_e$ is ongoing but $\partial\beta_e/\partial c_s$ is not continuous [4,5,11,14].

The equilibrium value $\beta_e$ for a system is given by minimizing $E(\beta,\gamma)$ with respect to $\beta$, i.e. imposing the following conditions as [4, 5]

$$\frac{\partial E(\beta)}{\partial \beta} = 0, \tag{17}$$

The equilibrium values $\beta_e$ for the $U^{BF}(5) \leftrightarrow O^{BF}(6)$ transition for even-even and boson-fermion systems with the different angular moments are calculated as

$$\beta_e^{IBM-1}(c_s,c_d) = \pm\sqrt{-\frac{g(N_B-1)(c_sc_d-c_d^2)+1.2\delta+4\gamma+\alpha(c_d^2-c_s^2)}{g(N_B-1)(c_d^2-c_sc_d)+1.2\delta+4\gamma+\alpha(c_d^2-c_s^2)}} \tag{18}$$

$$\beta_e^{IBFM-1,j=1/2}(c_s,c_d) = \pm\sqrt{-\frac{g(N_B-1)(c_sc_d-c_d^2)+1.2(\delta+\eta)+4\gamma+\alpha(c_d^2-c_s^2)}{g(N_B-1)(c_d^2-4c_sc_d+3c_s^2)-2.4(\delta+\eta)-8\gamma-2\alpha(c_d^2-c_s^2)}} \tag{19}$$

$$\beta_e^{IBFM-1,j=3/2}(c_s,c_d) = \pm\sqrt{-\frac{g(N_B-1)(c_sc_d-c_d^2)+1.2\delta+10\gamma+\alpha(c_d^2-c_s^2)}{g(N_B-1)(c_d^2-c_sc_d)+1.2\delta+10\gamma+\alpha(c_d^2-c_s^2)}} \tag{20}$$

$$\beta_e^{IBFM-1,j=1/2,3/2,5/2}(c_s,c_d) = \pm\sqrt{-\frac{g(N_B-1)(c_sc_d-c_d^2)+4\delta+24(\eta+\gamma)+\alpha(c_d^2-c_s^2)}{g(N_B-1)(c_d^2-c_sc_d)+4\delta+24(\eta+\gamma)+\alpha(c_d^2-c_s^2)}} \tag{21}$$
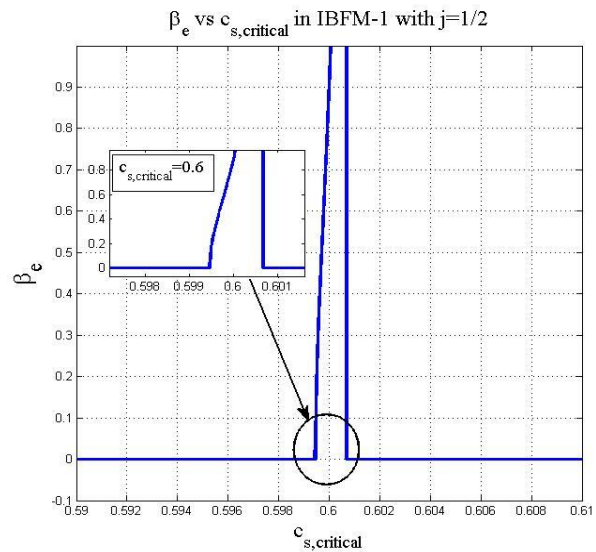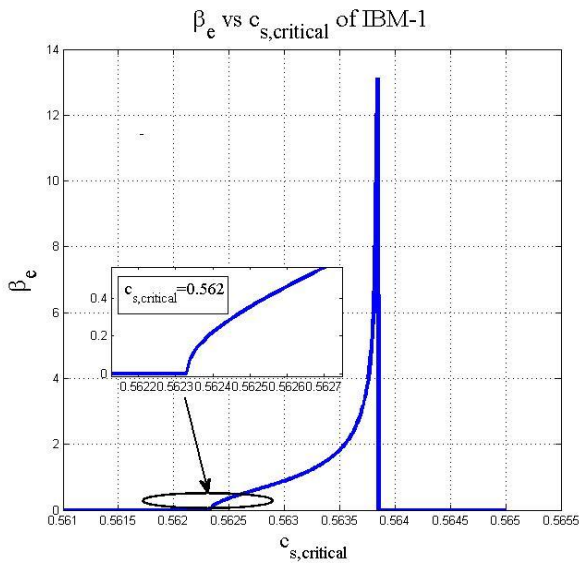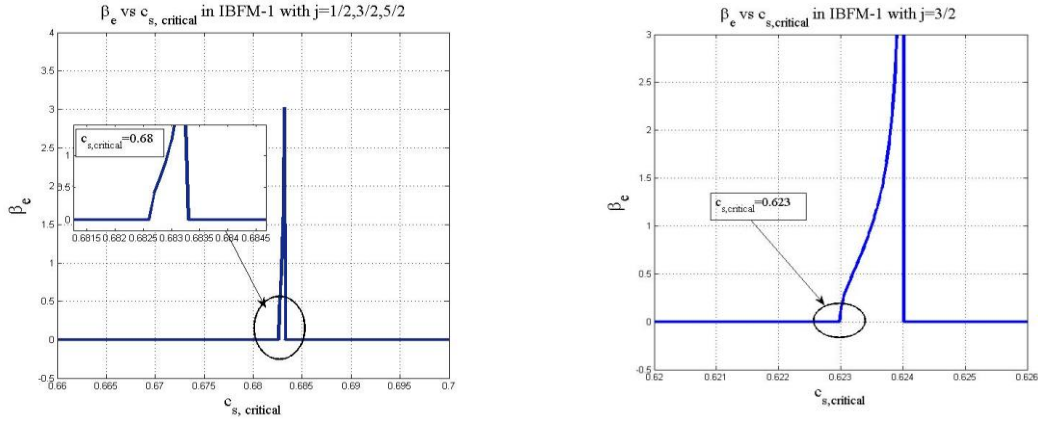
**Fig.8** Evolution of the equilibrium deformation parameter $\beta_e$ in the even-even and Odd-A systems as a function of the control

parameter $c_s$ along with the U(5)-SO(6) transition region.

The equilibrium values for the $U^{BF}(5) \leftrightarrow O^{BF}(6)$ transition in even-even nuclei and odd-A in a single nucleon in j = 1/2, j = 3/2 and j = 1/2, 3/2, 5/2 orbits are shown in Fig. 8. From this figure, one can see the effects of the odd particle at the critical point of phase transition that remain approximately constant $c_s < c_{s,critical}$ and only begin to change rapidly $c_s > c_{s,critical}$ . The critical point is approximately the same location as the calculated surface energy.
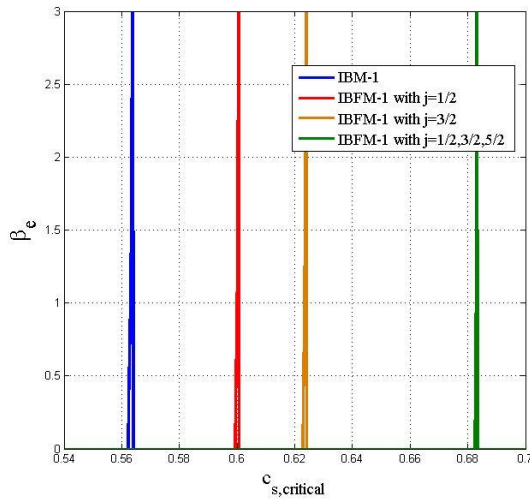


**Fig.9** The evolution of the equilibrium deformation parameter along with the shape phase transition for the boson core and the odd-even systems considering different angular momenta

We can see in Fig. (9) that the modification induced by the presence of a fermion shifts the location of the critical point. The critical point position in odd-A nuclei moves considerably to the right of the crucial point position in even-even nuclei actually, the single fermion can be considered as a catalyst. Phase transitions from the spherical to the $\gamma$-unstable limit for odd nuclei in the frameworks of the IBFM and the Bohr collective model, with the bizarre nucleon lying in a $j = 3/2$ shell, were considered in Refs. [6-13]. So, it must be, valuable and worthwhile to compare the present method and results to the method and results of these papers. The paper by Alonso *et al.* Refs.[6,10] investigated the phase transition around the critical point in the evolution from spherical to $\gamma$ -unstable shapes in the cases where an odd $j = 3/2$ and $j = 1/2$ $3/2\ and\ 5/2$ particle coupled to an even-even boson core that undergoes a transition from the spherical U(5) to the $\gamma$ -unstable O(6) situation. They used the coherent state formalism and semiclassical approach to get energy eigenvalues. We also investigated the transition from the spherical to the $\gamma$ -unstable limit in odd-*A* nuclei when only the boson core experiences the transition and fermions $j = 1/2\ 3/2\ 5/2$ are coupled to the boson core. In our work, the eigenstates and energy eigenvalues for the

transitional region were evaluated by using the dual algebraic structure, the Richardson-Gaudin method, and affine SU(1,1) Lie algebra; thus, the natures of the two schemes are entirely different. The E(5|12) algebraic solution for A-odd nuclei is introduced using the dual algebraic structures and the affine SU(1,1) Lie Algebra within the framework of the IBFM. Our results are like the results of Refs. [6-13] that the coupling of the single fermion with angular momentum j to the even-even system doesn't change the geometry imposed by the boson core performing the transition, and only the position of the critical points has been shifted by the addition of the odd particle with respect to the even case.

## VI. Conclusions

In this paper, the effect of the coupling of a single fermion with a boson core that performs a transition from spherical to $\gamma$-unstable shapes, in particular, at the critical point is investigated. The E(5|12) algebraic solution for A-odd nuclei is introduced using the dual algebraic structures and the affine SU(1,1) Lie Algebra within the framework of the interacting boson-fermion model. The energy surfaces and equilibrium values $\beta_e$ along the shape phase transition are calculated for the boson core and the odd-A systems considering different angular momenta j: 1/2, 3/2, and 5/2 orbits. We have presented an analysis of QPT in a system of $N_B$ bosons and one fermion and shown that adding a fermion significantly modifies the critical value at which the phase transition occurs. We found that the coupling of the single fermion with angular momentum j to the even-even system doesn't change the geometry imposed by the boson core performing the transition. Only the position of the critical point has been shifted by the addition of the

odd particle with respect to the even case. So, -known properties in the even-even nuclei also persist in the odd-A system.

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